M20 – 4.4 The Quadratic Formula

Key Ideas

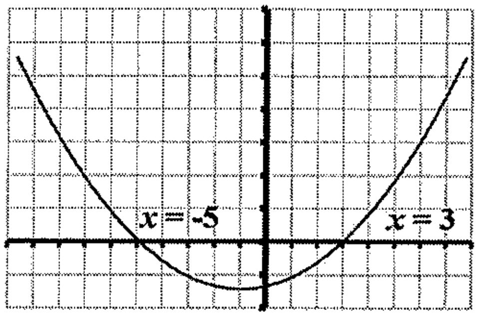
Solving a quadratic equation

$$x^2+2x-15=0$$

Method 2: Factor

$$(x-3)(x+5)=0$$

Method 1: Graph



Method 3: Quadratic Formula

$$ax^2+bx+c=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

While in Standard Form, $y = ax^2 + bx + c$, the zeroes of the function can be solved using the Quadratic Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This gives you two possible solutions:

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Use this to solve for the x-intercepts of a quadratic function that cannot be factored easily.

Note: Discriminate: The value under the root, $b^2 - 4ac$

- If the discriminate is greater than 0, there are two x-intercepts.
- If the discriminate is equal to 0, there is one x-intercept.
- If the discriminate is less than 0, and cannot be square-rooted, there are no x-intercepts.

Math 20-1 KEY

Part 1 – Derivation of the Quadratic Formula (2 marks out of 12 on next SQ; Marked with Rubric)

Step #1: Switch to Vertex Form

$$y = ax^2 + bx + c$$

$$y = (ax^2 + bx) + c$$

$$y = a\left(x^2 + \frac{b}{a}x\right) + c$$

$$y = a\left(x^2 + \frac{b}{2a}x + \frac{b}{2a}x\right) + c$$

$$y = a\left(x^2 + \frac{b}{2a}x + \frac{b}{2a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a}$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Step #2: Solve for the Zeroes

$$0 = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

$$\frac{b^2}{4a} - c = a\left(x + \frac{b}{2a}\right)^2$$

$$\frac{b^2}{4a} - \frac{4ac}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

$$\frac{b^2 - 4ac}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

$$\frac{b^2 - 4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

$$\pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = x + \frac{b}{2a}$$

$$\pm\frac{\sqrt{b^2 - 4ac}}{2a} = x + \frac{b}{2a}$$

$$-\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = x$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = x$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Part 2 – When would we use it?

This is ideally used for a quadratic that isn't factorable.

Q1: Solve $x^2 + 13x + 40 = 0$ using each method.

Factoring

$$x^2+13x+40=0$$
$$(x+5)(x+8)=0$$
$$x+5=0 \qquad x+8=0$$
$$\boxed{x=-5} \qquad \boxed{x=-8}$$

Converting to Vertex Form

$$x^2+13x+40=0$$
$$(x^2+13x)+40=0$$
$$\left(x^2+\tfrac{13}{2}x+\tfrac{13}{2}x\right)+40=0$$
$$\left(x^2+\tfrac{13}{2}x+\tfrac{13}{2}x+\tfrac{169}{4}\right)+40-\tfrac{169}{4}=0$$
$$\boxed{\left(x+\tfrac{13}{2}\right)^2-\tfrac{9}{4}=0} \text{ Vertex Form}$$
$$+\tfrac{9}{4} \qquad +\tfrac{9}{4}$$
$$\left(x+\tfrac{13}{2}\right)^2=\tfrac{9}{4}$$
$$x+\tfrac{13}{2}=\sqrt{\tfrac{9}{4}}$$
$$x+\tfrac{13}{2}=+\tfrac{3}{2} \qquad x+\tfrac{13}{2}=-\tfrac{3}{2}$$
$$-\tfrac{13}{2} \quad -\tfrac{13}{2} \qquad -\tfrac{13}{2} \quad -\tfrac{13}{2}$$
$$\boxed{x=-5} \qquad \boxed{x=-8}$$

Quadratic Formula

$$1x^2+13x+40=0$$
$$a=1 \quad b=13 \quad c=40$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-(13)\pm\sqrt{13^2-4(1)(40)}}{2(1)}$$
$$x=\frac{-13\pm\sqrt{169-160}}{2}$$
$$x=\frac{-13\pm\sqrt{9}}{2}$$
$$x=\frac{-13\pm3}{2}$$
$$x=\frac{-13+3}{2} \qquad x=\frac{-13-3}{2}$$
$$\boxed{x=-5} \qquad \boxed{x=-8}$$

Q2: Solve $x^2 + 4x - 4 = 0$ using each method

Factoring

$$x^2+4x-4=0$$
$$\square+\square=4$$
$$\square\times\square=-4$$

Not factorable!

Converting to Vertex Form

$$x^2+4x-4=0$$
$$(x^2+4x)-4=0$$
$$(x^2+2x+2x)-4=0$$
$$(x^2+2x+2x+4)-4-4=0$$
$$\boxed{(x+2)^2-8=0} \text{ Vertex Form}$$
$$+8 \quad +8$$
$$(x+2)^2=8$$
$$x+2=\sqrt{8}$$
$$x+2=+\sqrt{8} \qquad x+2=-\sqrt{8}$$
$$-2 \quad -2 \qquad -2 \quad -2$$
$$\boxed{x_1=\sqrt{8}-2 \approx 0.83} \qquad \boxed{x_2=-\sqrt{8}-2 \approx -4.83}$$

Quadratic Formula

$$1x^2+4x-4=0$$
$$a=1 \quad b=4 \quad c=-4$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-4\pm\sqrt{4^2-4(1)(-4)}}{2(1)}$$
$$x=\frac{-4\pm\sqrt{32}}{2}$$
$$x=\frac{-4\pm4\sqrt{2}}{2}$$
$$\boxed{x_1=-2+2\sqrt{2} \approx 0.83} \qquad \boxed{x_2=-2-2\sqrt{2} \approx -4.83}$$

32, 2, 16, 2, 8, 2, 2

Part 3 – The Discriminate

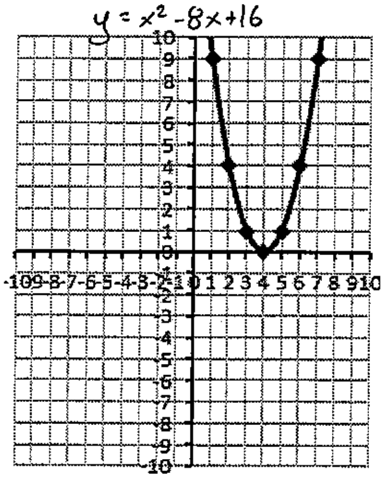
Note: Discriminate: The value under the root, $b^2 - 4ac$

- If the discriminate is greater than 0, there are two x-intercepts.
- If the discriminate is equal to 0, there is one x-intercept.
- If the discriminate is less than 0, and cannot be square-rooted, there are no x-intercepts.

Q3: For each of the following functions, determine the discriminate, and how it relates to the number of zeroes.

$x^2 - 8x + 16 = 0$

$y = x^2 - 8x + 16$



$b^2 - 4ac$

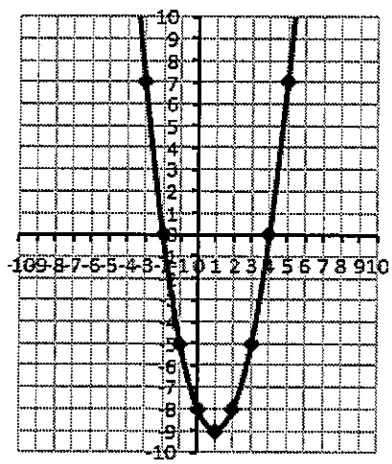
$(-8)^2 - 4(1)(16)$

$64 - 64$

0

Discriminate $= 0$
so only one "zero"

$x^2 - 2x - 8 = 0$



$b^2 - 4ac$

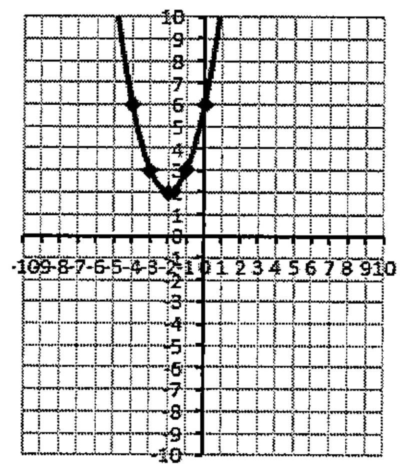
$(-2)^2 - 4(1)(-8)$

$4 + 32$

40

Discriminate > 0
so two distinct "zeroes"

$x^2 + 4x + 6 = 0$



$b^2 - 4ac$

$(4)^2 - 4(1)(6)$

$16 - 24$

-8

Discriminate < 0
so no "zeroes"

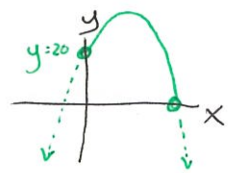
Part 4 – Examples from Word Problems

Use the following information to answer Q4-Q5:

An object is shot vertically off the edge of a 20m tall cliff at 4m/s [up]. The height of the object, as a function of time, is given by the equation

$$h(t) = -5t^2 + 4t + 20$$

where *h(t)* is measured in meters, and *t* is measured in seconds.



Q4: Use the quadratic equation to determine how long it takes the object to reach the ground.

$a = -5 \quad b = 4 \quad c = 20$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-4 \pm \sqrt{4^2-4(-5)(20)}}{2(-5)}$$

$$x = \frac{-4 \pm \sqrt{16+400}}{-10}$$

$$x = \frac{-4 \pm \sqrt{416}}{-10}$$

$$x_1 = \frac{-4+\sqrt{416}}{-10} \doteq -1.64$$

$$x_2 = \frac{-4-\sqrt{416}}{-10} \doteq +2.44$$

This one makes sense.

$\boxed{\text{Takes 2.44 sec to hit the ground}}$

Q5: Convert the function to Vertex Form. Determine the maximum height of the object.

$$h(t) = (-5t^2+4t)+20$$

$$= -5\left(t^2 - \tfrac{4}{5}t\right) + 20$$

$$= -5\left(t^2 - \tfrac{2}{5}t - \tfrac{2}{5}t\right) + 20$$

$$= -5\left(t^2 - \tfrac{2}{5}t - \tfrac{2}{5}t + \tfrac{4}{25}\right) + 20 + \tfrac{4}{5}$$

$$\boxed{h(t) = -5\left(t - \tfrac{2}{5}\right)^2 + \tfrac{104}{5}}$$ Vertex form

Vertex occurs at (t, h) is $\left(\tfrac{2}{5}, \tfrac{104}{5}\right)$.

$\boxed{\text{So maximum height is } \tfrac{104}{5} \text{ m, or } 20.8\text{m.}}$